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Localized tachyons and the quantum McKay correspondence

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Abstract: The condensation of closed string tachyons localized at the fixed point of a $\mathbb{C}^d/\Gamma$ orbifold can be studied in the framework of renormalization group flow in a gauged linear sigma model. The evolution of the Higgs branch along the flow describes a resolution of singularities via the process of tachyon condensation. The study of the fate of D-branes in this process has lead to a notion of a "quantum McKay correspondence." This is a hypothetical correspondence between fractional branes in an orbifold singularity in the ultraviolet with the Coulomb and Higgs branch branes in the infrared. In this paper we present some nontrivial evidence for this correspondence in the case $\mathbb{C}^2/\mathbb{Z}_n$ by relating the intersection form of fractional branes to that of "Higgs branch branes," the latter being branes which wrap nontrivial cycles in the resolved space.

Keywords: Superstrings and Heterotic Strings, D-branes, Tachyon Condensation, Differential and Algebraic Geometry.

Contents

1. Introduction and summary

In [1] Adams, Polchinski, and Silverstein introduced the study of localized tachyon condensation in closed string theories with target space of the form $M \times \mathbb{C}^d/\Gamma$ where M is Minkowski space and the orbifold action by a discrete subgroup Γ of the rotation group breaks supersymmetry. These models have turned out to provide a rich playground for studying both closed string tachyon condensation as well as the behavior of branes in the presence of closed string renormalization group (RG) flow. The present paper continues the study of D-branes in these models. We will refine a proposal of [2] (reviewed below) for resolving a paradox related to "missing D-brane charge."

Although the basic techniques should generalize to any d, and abelian Γ, we focus primarily on the orbifold $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ defined by the identification

$$(Z_1, Z_2) \sim (\omega Z_1, \omega^p Z_2) \tag{1.1}$$

where $\omega = e^{2\pi i/n}$ and we assume $(n,p) = 1$. Of course, string theory contains spacetime fermions so we must actually lift the orbifold group $\Gamma \cong \mathbb{Z}_n$ to a subgroup of the spin group $\hat{\Gamma} \subset \mathrm{SU}(2)\times\mathrm{SU}(2)$. The lift $\hat{\Gamma}$ depends on p mod $2n$. We will take $p \in (-n,n)$. As we review in section 2 below, because we focus on the type-II string in this paper, p must be odd. We will need to impose further constraints on n,p. These will be discussed momentarily.

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When $\mathbb{Z}_{n(p)}$ does not preserve supersymmetry, i.e. when Γ is not a subgroup of $\mathrm{SL}(2,\mathbb{C})$, then there are localized tachyons in the orbifold. The condensation of these tachyons can be described in several ways. The original approach of [1] was to use D-brane probes and supergravity analysis. One can also use techniques suited to the study of RG flow in $\mathcal{N} = (2,2)$ 2d quantum field theories [3, 4, 5, 6, 2, 7]. Some basic aspects of these techniques are reviewed in section 2 below. The picture which emerges is that there is an elaborate set of possible flows, and the infrared (IR) limit of the RG flow is a spacetime that blows up into a (partial) resolution of the singularity, producing far separated "islands of supersymmetric vacua." The most complete picture thus far was obtained in [2] where it is shown that the perturbation by a generic linear combination of (primitive) chiral ring operators flows (sufficiently far into the IR) to a Higgs branch of vacua described by the minimal, or Hirzebruch-Jung resolution $\mathcal{X}$ of the singularity $\mathbb{C}^2/\mathbb{Z}_{n(p)}$. As explained at length in [3, 2, 6] this complex manifold is a generalization of the familiar ALE resolution of $\mathbb{C}^2/\mathbb{Z}_{n(-1)}$ orbifolds. In particular the origin of $\mathbb{C}^2$ is blown up into a collection of exceptional divisors, $\Sigma_\alpha \cong \mathbb{C}P^1$, with intersection matrix

$$-C = \begin{pmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ 1 & -a_2 & 1 & \cdots & 0 \\ 0 & 1 & -a_3 & \cdots & 0 \\ \cdots & \cdots & \cdots\cdots & & \vdots \\ 0 & 0 & 0 & \cdots & -a_r \end{pmatrix} \tag{1.2}$$

where the integers $a_\alpha \geq 2$ are the partial quotients in the continued fraction expansion

$$\frac{n}{p_1} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \frac{1}{\cdots 1/a_r}}} := [a_1, \ldots, a_r]. \tag{1.3}$$

where $p_1 = p$, $p > 0$ and $p_1 = p + n$, $p < 0$. For each successive string of 2's in the partial quotients there is a supersymmetric island. All of the flows discussed in [1, 3, 4, 7] are special cases of this prescription. In this paper, we restrict attention to the type-II string. Therefore, we must take care that the generators of the chiral ring are not projected out by the GSO projection. In section 2 [equations (2.15)–(2.19)], we show that the requirement that the generators of the chiral ring survive the GSO projection is equivalent to the statement that p is negative and that the partial quotients a_α in (1.3) are all even. This is the further restriction on n, p alluded to above.

It was pointed out in [2] that the above description of the RG flows leads to a paradox concerning D-brane charge (more generally, concerning the category of topological D-branes). At the orbifold point the boundary states form a lattice, which may be identified [8, 9, 10] with the equivariant K-theory lattice $K_{\mathbb{Z}_n}(\mathbb{C}^2) \cong \mathbb{Z}^n$. However, the above description of the resolved space suggests that the charges of boundary states should span the lattice $K^0(\mathcal{X}) \cong \mathbb{Z}^{r+1}$. Since $r' := n - 1 - r > 0$, except in the supersymmetric case, we are left with a case of "missing brane charge." The resolution proposed in [2] is that the RG flow should be formulated in terms of the gauged linear sigma model (GLSM) [11, 12] and that one must account for branes in *all* the quantum vacua in the IR. In particular, one must take into account both the Higgs branch $\mathcal{X}$ as well as the Coulomb branch of vacua. In the present paper we shed more light on the relation between the fractional branes of the orbifold conformal field theory and the Coulomb and Higgs branch branes. We will perform a nontrivial check on the overall picture by studying the intersection form on boundary states, showing how that of the fractional branes "contains" the intersection form for Higgs branch branes.

There is a natural basis for the D-brane boundary states of the orbifold $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ which is in 1-1 correspondence with the unitary irreps ρ_a of $\mathbb{Z}_n$ [8]. We take $\rho_a(\omega) = \omega^a$. We denote the corresponding boundary states by e_a, $a = 0, \dots, n-1$. The regular representation $e_0 + \dots + e_{n-1}$ corresponds to a $D0$ brane, which may freely move away from the orbifold point, while $e_1, \dots, e_{n-1}$ correspond to "fractional branes," which are pinned at the orbifold point. The main result of this paper is that one must make a nontrivial change of basis from e_a to form boundary states $h_0, \dots, h_r$ and $c_1, \dots, c_{r'}$ corresponding to branes in the Higgs and Coulomb vacua, respectively. As explained in [2], and reviewed in section 3 below, the GLSM naturally makes a distinction between two sets of irreps of $\mathbb{Z}_n$. The "special representations" may be identified with a single $D2$ brane wrapping an exceptional divisor.[1] In this paper we refer to the "special representations" as the "Higgs representations," while the r' nontrivial irreducible representations in the complement will be called "Coulomb representations." Denoting Higgs representations by $\alpha = 1, \dots, r$ and Coulomb representations by $\nu = 1, \dots, r'$ we have

$$\begin{aligned} h_0 &= e_0 + \dots + e_{n-1} \\ h_\alpha &= e_\alpha + \sum_{\nu=1}^{r'} u_\alpha^{\ \nu} e_\nu \\ c_\nu &= e_\nu \end{aligned} \tag{1.4}$$

where $u_\alpha^{\ \nu}$ is a matrix of integers.

The main tool we employ to arrive at (1.4) is the intersection form on boundary states. Quite generally, if a, b are D-brane boundary conditions then the Witten index in the open string channel $\mathcal{H}_{ab}$ defines a bilinear pairing $\mathcal{B} \times \mathcal{B} \to \mathbb{Z}$, where $\mathcal{B}$ is the set of boundary conditions [13, 14, 15]. (We may think of these as objects in an additive category, or as a lattice of boundary states. If the ground ring is the complex numbers then the pairing is sesquilinear.) For example, if a, b define K-theory classes on a smooth spacetime then $(a, b) = \mathrm{Ind}(\not{D}^+_{\bar{a}\otimes b})$ is given by the index of the chiral Dirac operator.[2] Since the pairing is an index it is expected to be invariant under (finite) RG flow. Thus, comparison of the pairing of fractional branes and Higgs branch branes leads to a nontrivial relation between these branes.

The quadratic form on fractional branes is easily computed (see section 4 below). Let ρ_a, ρ_b be representations of Γ corresponding to fractional branes a, b, and let π_* be the

[1] Thanks to the sequence $0 \to \mathcal{O}(-\Sigma) \to \mathcal{O} \to \iota_*(\mathcal{O}_\Sigma) \to 0$, the special representations are also in correspondence with boundstates of a single $D4$ brane wrapping $\mathcal{X}$ and bound to a $D2$ brane wrapping one of the exceptional divisors Σ.

[2] Our pairing will always be symmetric or antisymmetric. In the literature on topological branes a different, asymmetric, pairing on boundary states is sometimes employed. The difference between the two pairings is analogous to (and sometimes, is) the difference between the Dirac index and the $\bar{\partial}$ index.

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projection of a (virtual) representation onto the trivial representation. Since Γ is lifted to $\hat{\Gamma}$ in the spin group, the chiral spin representations $S^{\pm}$ restrict to representations of Γ. The intersection form is given by

$$(a,b) = \pi_* \left(\bar{\rho}_a \otimes \rho_b \otimes (S^+ - S^-) \right). \tag{1.5}$$

For the case $d = 2$, $\Gamma = \mathbb{Z}_n$ this reduces to

$$I = \mathcal{S}^{(p-1)/2} + \mathcal{S}^{-(p-1)/2} - \mathcal{S}^{(p+1)/2} - \mathcal{S}^{-(p+1)/2} \tag{1.6}$$

where $\mathcal{S}$ is the $n \times n$ shift matrix. Equation (1.6) is to be contrasted with the intersection form for Higgs branch branes, the latter being the geometrical intersection form (1.2). The main technical step, which is explained in detail in section five below, is to find a matrix of *integers* $u_\alpha^{\ \nu}$ so that the change of basis (1.4) brings I to the block diagonal form

$$F = -UIU^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C' \end{pmatrix} \tag{1.7}$$

where C is defined in (1.2).

As explained in [2] the relation of fractional branes to the Coulomb and Higgs branes of the GLSM suggests a generalization of the McKay correspondence which was termed the "quantum McKay correspondence." (See [2] for an extensive set of references to the McKay correspondence in the math and physics literature.) The name "special representation" indeed originates from mathematical investigations into the generalization to non-crepant (spacetime non-supersymmetric) resolutions of the two-dimensional McKay correspondence [16, 17, 18]. In the math literature it is shown that there is a 1-1 correspondence between special representations and the tautological line bundles on $\mathcal{X}$. As explained in [2] and section 3 below, this is naturally understood in the context of the GLSM. The important new point, which is suggested by physics, is that by including the Coulomb branes one has a correspondence more analogous to the crepant case. Moreover, physics strongly suggests that the correspondence can be phrased at the level of derived categories of sheaves, and that it should apply to all toric resolutions of orbifold singularities. Some mathematical results relevant to this conjecture were announced in [19] and attributed to Orlov, but unfortunately no details have as yet been available.

We must stress an important technical restriction on our result. We have only managed to find a change of basis bringing (1.6) to the form (1.7) in the case where the GLSM can be embedded in the GSO-projected type-II string. As we have mentioned, this requires $p \in (-n, n)$ to be odd and negative and the partial quotients a_α to be even. We comment on the type-0 string in section 6. It is not clear to us if the restriction to the type-II string is of fundamental importance, or if it is merely a technical simplification. By studying specific examples one can show that an analogous change of basis in the type-0 string will in general be much more complicated than (1.4). It is perhaps important to note in this context that the type-0 branes are sources of the bulk tachyon.

To summarize, this paper is organized as follows. In the next section we briefly review the condensation of closed string tachyons in the twisted sectors of $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ orbifolds and show that the ring generators survive the GSO projection if and only if p is negative and all a_α are even integers. In section 3 we recall the gauged linear sigma model description of tachyon condensation and give a preview of what happens to the orbifold fractional branes, following [2]. In particular, we explain how the Higgs representations can be inferred from the $\mathbb{Z}_n$ action. In section 4 we study open strings living on the orbifold and compute the intersection matrices of the fractional branes. The main technical work is in section 5, where we explain how one can bring I to block-diagonal form (1.7). We show that this is possible precisely when the chiral ring generators are not projected out by the chiral GSO. We discuss type-0 theory, and some other loose ends, in section 6.

2. Chiral ring, singularity resolution and GSO projection

We consider superstring theory in 9+1 dimensions. The orbifolding by $\mathbb{Z}_{n(p)}$ happens in the 67 and 89 planes, parametrized by complex coordinates $Z^{(1)}$ and $Z^{(2)}$. The orbifold group is generated by

$$w = \exp\left(\frac{2\pi i}{n}(J_{67} + pJ_{89})\right), \tag{2.1}$$

where J_{67} and J_{89} generate rotations in two complex planes and p is defined $\mathrm{mod}\, 2n$. We will take the fundamental domain to be $p \in (-n, n)$. The action of w^n on the Ramond sector ground state is a multiplication by $(-1)^{p\pm 1}$, depending on chirality. When p is even, this acts as $(-1)^F$ where F is the spacetime fermion number. In type-II, there is no bulk tachyon and there are closed string fermions in the bulk, hence p must be odd [1].

Let us briefly review the formulation in NSR formalism. For a more complete discussion see [3]. The complex coordinate $Z^{(i)}$ is promoted to the worldsheet chiral superfield

$$\Phi^{(i)} = Z^{(i)} + \theta_+\psi_+^{(i)} + \theta_-\psi_-^{(i)} + \cdots . \tag{2.2}$$

Here pluses and minuses in the subscript correspond to the right and left movers; we reserve bar for the target space complex conjugation. The fermions can be bosonised

$$\psi_+^{(i)} = e^{iH_+^{(i)}}; \qquad \psi_-^{(i)} = e^{iH_-^{(i)}} . \tag{2.3}$$

The orbifold action on the worldsheet fields is

$$w: \; \Phi^{(1)} \to e^{\frac{2\pi i}{n}}\Phi^{(1)} \qquad \Phi^{(2)} \to e^{\frac{2\pi i p}{n}}\Phi^{(2)} . \tag{2.4}$$

The theory contains $n-1$ twisted sectors, labeled by $s = 1, 2, \ldots, n-1$. We want to construct vertex operators X_s that correspond to ground states in the twisted sectors. A useful ingredient is an operator

$$X_s^{(i)} = \sigma_{s/n}^{(i)} \exp\left[i\left(\frac{s}{n}\right)(H_+^{(i)} - H_-^{(i)})\right]; \qquad s = 1, 2, \ldots, n-1 \tag{2.5}$$

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where $\sigma_{s/n}$ is the bosonic twist s operator [20]. In the following we will restrict our attention to the right movers and will drop the $+$ and $-$ subscripts. The tachyon vertex operator can now be written as

$$X_s = X_s^{(1)} X^{(2)}_{n\{\frac{sp}{n}\}}, \qquad (2.6)$$

where $\{x\} \equiv x - [x]$ is the fractional part of x. The generators of the chiral ring W_α, $\alpha = 1, \ldots, r$ form a collection of (in general) relevant operators. Turning these on in the action induces RG flow to the minimal resolution of the singularity [3, 2]. For the $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ orbifold such a resolution is encoded in the continued fraction

$$\frac{n}{p_1} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \ldots}} := [a_1, a_2, \ldots, a_r], \qquad (2.7)$$

where

$$p_1 = p, \quad p > 0; \qquad p_1 = p + n, \quad p < 0, \qquad (2.8)$$

defines the action of the orbifold group on the bosonic fields; $p_1 \in (0, n)$. The generators of the chiral ring are in one-to-one correspondence with the $\mathbb{P}^1$'s of the minimal resolution, which requires r $\mathbb{P}^1$'s with self-intersection numbers $-a_\alpha$. The intersection number of adjacent $\mathbb{P}^1$'s is equal to one. In other words, the intersection matrix is the negative of the generalized Cartan matrix,

$$C_{\alpha\beta} = -\delta_{\alpha,\beta-1} + a_\alpha \delta_{\alpha,\beta} - \delta_{\alpha,\beta+1}. \qquad (2.9)$$

It is possible to label the generators of the chiral ring by the set of integers $\{q_i\}$ and $\{p_i\}$ which are determined by the continued fraction $[a_1, a_2, \ldots, a_r]$ via the recursion relations [2]:

$$\begin{aligned} \frac{p_{j-1}}{p_j} &= [a_j, a_{j+1}, \ldots, a_r] \\ \frac{q_{j+1}}{q_j} &= [a_j, \ldots, a_1], \qquad 1 \le j \le r \end{aligned} \qquad (2.10)$$

with the initial conditions $q_0 = 0$, $q_1 = 1$, $p_{r+1} = 0$, $p_r = 1$. The vectors

$$\frac{1}{n}\mathbf{v}_j = \frac{1}{n}(q_j, p_j), \qquad j = 0, \ldots, r+1 \qquad (2.11)$$

satisfy the recursion relations

$$a_j \mathbf{v}_j = \mathbf{v}_{j-1} + \mathbf{v}_{j+1}, \qquad 1 \le j \le r, \qquad (2.12)$$

In fact, $\mathbf{v}_\alpha$, $\alpha = 1, \ldots, r$, are in one-to-one correspondence with the vectors providing the minimal resolution of the singular fan describing $\mathbb{C}^2/\mathbb{Z}_{n(p)}$. They also label the generators of the chiral ring

$$W_\alpha = X^{(1)}_{q_\alpha} X^{(2)}_{p_\alpha}. \qquad (2.13)$$

In other words, q_j and p_j are the charges of $\mathrm{U}(1)_R$ symmetries generated by the currents $\psi^{(i)}\bar{\psi}^{(i)}$, $i = 1, 2$. The intersection matrix (2.9) encodes the ring relations between the generators [3]

$$W_{j-1} W_{j+1} = W_j^{a_j}, \qquad 1 \le j \le r. \qquad (2.14)$$

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In type-II there is a subtlety having to do with the existence of chiral GSO projection. By comparing with the spectrum of the light-cone Green-Schwarz formulation of the orbifold, one finds that the action of $(-1)^F$ is given by [3]

$$H^{(1)} \to H^{(1)} + \pi p \,, \qquad H^{(2)} \to H^{(2)} - \pi \,. \tag{2.15}$$

which implies that p must be odd.[3] The action on tachyon vertex operators is

$$X_s \to (-1)^{\frac{[sp}{n]}} X_s \,. \tag{2.16}$$

Only odd operators in the $(-1,-1)$ picture survive chiral GSO projection, which means that all X_s with even $[sp/n]$ are projected out, while those with odd $[sp/n]$ survive. This means that for positive p at least one generator of the chiral ring, $W_1 = X_1$, is projected out. In the following we will be interested in the examples where the set of generators is left intact by the GSO projection. A necessary condition is therefore that p must be negative, which implies $p_1 = p + n$.

According to (2.13), (2.15), the action of $(-1)^F$ on the generators is

$$W_j \to (-1)^{B'_j} W_j \,, \tag{2.17}$$

where

$$B'_j = \frac{1}{n}(p_j - p q_j) = B_j - q_j \,. \tag{2.18}$$

Here B_j are integers which satisfy the same recursion relations as q_j and p_j [2]

$$a_j B_j = B_{j-1} + B_{j+1} \,, \qquad 1 \le j \le r \tag{2.19}$$

with initial conditions $B_0 = 1$, $B_1 = 0$. From (2.12), (2.18) and (2.19) it follows that B'_j satisfy the same recursion relations (2.19) with boundary conditions $B'_0 = 1, B'_1 = -1$. According to (2.17), all W_j survive the GSO projection if and only if B'_j are all odd, which is equivalent to all a_j being even. As we will see below, precisely for such orbifolds the intersection matrix of fractional branes contains the intersection matrix of the minimal cycles in the Hirzebruch-Jung resolution.

3. GLSM and special representations

In [2] the gauged linear sigma model was used to shed light on the fate of fractional D-branes in the process of twisted tachyon condensation. The basic puzzle that was adressed in [2] is the following. The charges of fractional D-branes on the orbifold are described by the equivariant K-theory, which is isomorphic to $\mathbb{Z}^n$ as a $\mathbb{Z}$-module. On the other hand, K-theory of the Hirzebruch-Jung manifold is of rank $r+1$. The compactly supported K-theory is generated by r branes wrapping a basis of two-cycles, together with the $D0$ brane. What happens to the other $r' = n - r - 1$ branes? The gauged linear sigma model answer [2] is that these extra branes live on the Coulomb branch of the theory, while it is the Higgs branch which describes the geometry of the resolved space.

[3]The GSO projection is also discussed in [4, 21, 22].

The $D2$ branes wrapping the r independent cycles in $\mathcal{X}$ correspond to r "special representations" of the orbifold group $\mathbb{Z}_n$. Let us now recall the special representations of the orbifold group and how they appear. The gauged linear sigma model description [2] involves r U(1) gauge fields A_α, $\alpha = 1 \dots r$ and $r+2$ chiral fields X_i, $i = 0 \dots r+1$ with charges

$$Q_{i\alpha} = \delta_{i,\alpha-1} - a_\alpha \delta_{i,\alpha} + \delta_{i,\alpha+1} \,. \tag{3.1}$$

There are also r FI terms ζ_α, which are renormalized. The geometry for a given set of ζ_α's is determined by solving F and D-term equations and modding out by the unbroken gauge group. At the UV fixed point, $\zeta \to -\infty$, this unbroken gauge group is precisely $\mathbb{Z}_n$ and the theory is reduced to the $\mathbb{C}^2/\mathbb{Z}_n$ orbifold. In the IR, $\zeta \to +\infty$, the RG flow leads to the Hirzebruch-Jung space; the non-minimal curves are blown down [2]. This space can be viewed as a quotient of a $\mathrm{U}(1)^r$ bundle S_ζ over the base space $\mathcal{X}$ by the natural action of the $\mathrm{U}(1)^r$ group element $g = (e^{i\theta_1} \dots e^{i\theta_r})$,

$$g : X_i \to e^{iQ_{i\alpha}\theta_\alpha} X_i \,. \tag{3.2}$$

The K-theory of $\mathcal{X}$

$$K^0(\mathcal{X}) = \mathbb{Z} \oplus \mathbb{Z}^r \,. \tag{3.3}$$

is generated by $\mathcal{O}$, the trivial line bunlde on $\mathcal{X}$, and tautological line bundles R_α corresponding to D-branes filling $\mathcal{X}$ with magnetic monopoles in different exceptional divisors. These are the vector bundles associated to the representations $\rho_\alpha(g) = e^{i\theta_\alpha}$,

$$R_\alpha = (S_\zeta \times \mathbb{C}) / \mathrm{U}(1)^r \,, \tag{3.4}$$

where the action of $g \in \mathrm{U}(1)^r$ is given by (3.2) and

$$g : v \to e^{-i\theta_\alpha} v \,, \qquad v \in \mathbb{C} \,. \tag{3.5}$$

What happens to these D-banes as the FI parameters $\zeta_\alpha \to -\infty$, where $\mathcal{X}$ becomes a $\mathbb{C}^2/\mathbb{Z}_n$ orbifold? The representations ρ_α should now be restricted to the unbroken gauge group $\mathbb{Z}_n$. As explained in [2], this unbroken gauge group consists of elements

$$\left(e^{i\theta_1}, \dots, e^{i\theta_r}\right) = \left(\exp\left[2\pi i \frac{p_1 m_1}{n}\right], \dots, \exp\left[2\pi i \frac{p_r m_r}{n}\right]\right) \tag{3.6}$$

where $m_\alpha \in \mathbb{Z}$. Hence, R_α's correspond to special representations of $\mathbb{Z}_n$ labeled by p_α.

4. Intersection form for fractional branes of the $\mathbb{C}^d/\Gamma$ orbifold

Following [13, 14, 15] we define the brane intersection form on boundary conditions a, b to be:

$$I_{ab} = \mathrm{tr}_{R,ab}(-1)^F q^{L_0 - \frac{c}{24}} \,. \tag{4.1}$$

Here the trace is over the states of the open string suspended between D-branes which correspond to representations of Γ labeled by a and b and F is the worldsheet fermion number. The matrix I_{ab} is actually an index, i.e. does not depend on the modulus of the

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worldsheet cylinder q, and counts the difference of positive and negative chirality Ramond ground states.

The computation of I_{ab} for the spacetime supersymmetric $\mathbb{C}^d/\mathbb{Z}_n$ orbifolds can be found in [23]. We will describe here the computation along the same lines for $\mathbb{C}^2/\mathbb{Z}_{n(p)}$. We then give a simple general formula for $\mathbb{C}^d/\Gamma$. The cylinder amplitude that is invariant under the orbifold projection is

$$I_{ab} = \frac{1}{n} \sum_{g \in \mathbb{Z}_n} \mathrm{tr}_a(g)\, \mathrm{tr}_b(g)\, \mathrm{Tr}_R \left(g(-1)^F\, q^{L_0 - \frac{c}{24}} \right) \tag{4.2}$$

where $\mathrm{tr}_a(g)$ stands for the trace over the Chan-Paton factors; it is convenient to choose a basis where the action of g on Chan-Paton factors is diagonal [8]. Another ingredient,

$$\mathrm{Tr}_R \left(g\, q^{L_0 - \frac{c}{24}} \right) = \prod_{i=1,2} \mathrm{Tr}_{Z^{(i)}} \left(g\, q^{L_0 - \frac{c}{24}} \right) \mathrm{Tr}_{\psi^{(i)},R} \left(g\, (-1)^F\, q^{L_0 - \frac{c}{24}} \right) \tag{4.3}$$

can be evaluated in the NSR formalism separately for worldsheet bosons and fermions. The generator of the orbifold group w acts on the complex boson as

$$w:\; Z^{(i)} \to e^{2\pi i \nu_i} Z^{(i)}\,, \tag{4.4}$$

where $\nu_1 = 1/n$, $\nu_2 = p/n$. Hence,

$$\mathrm{Tr}_{Z^{(i)}} \left(g\, q^{L_0 - \frac{c}{24}} \right) = q^{-\frac{1}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i \nu_i} q^n)^{-1} (1 - e^{-2\pi i \nu_i} q^n)^{-1} \tag{4.5}$$

The fermion part can be evaluated in the similar manner. The nontrivial ingredient is now the presence of zero modes in the R sector. These zero modes satisfy anticommutation relations:

$$\{\psi_0^{(i)}, \bar{\psi}_0^{(j)}\} = \delta^{ij}\,, \qquad \{\psi_0^{(i)}, \psi_0^{(j)}\} = \{\bar{\psi}_0^{(i)}, \bar{\psi}_0^{(j)}\} = 0\,. \tag{4.6}$$

The ground state must furnish the representation of this algebra; it is a spacetime spinor labeled by the eigenvalues of $S^{(i)} = \bar{\psi}_0^{(i)} \psi_0^{(i)} - \frac{1}{2}$ which take values $s^{(i)} = \pm 1/2$. Then $g = w^s \in \mathbb{Z}_{n(p)}$ acts on the ground state as

$$g = w^s:\; |s^{(1)}, s^{(2)}\rangle \to \exp\left(\sum_{i=1,2} 2\pi i s^i \nu^i s \right) |s^{(1)}, s^{(2)}\rangle\,. \tag{4.7}$$

Note that p must be odd in order for the orbifold group to be $\mathbb{Z}_n$, $w^n = 1$. The fermion part of the trace is

$$\mathrm{Tr}_{\psi^{(i)}} \left(g(-1)^F\, q^{L_0 - \frac{c}{24}} \right) = 2 \sin(\pi \nu_i \alpha)\, q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i \nu_i} q^n)(1 - e^{-2\pi i \nu_i} q^n)\,. \tag{4.8}$$

Combining everything together, the intersection form is

$$I_{ab} = \frac{4}{n} \sum_{s=0}^{n-1} \exp\left(\frac{2\pi i (a-b) s}{n} \right) \sin\left(\frac{\pi s}{n} \right) \sin\left(\frac{\pi s p}{n} \right). \tag{4.9}$$

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Here the value of $g = w^s$ in the representation of $\mathbb{Z}_{n(p)}$ labeled by a is $e^{\frac{2\pi i a s}{n}}$. One observes that I_{ab} is not invariant under $p \to p \pm n$. This is not really surprising, as type-II theory is not invariant under such a shift.[4] The intersection form (4.9) can be easily evaluated; the result is

$$I_{ab} = \delta_{a-b-\frac{1-p}{2}} + \delta_{a-b+\frac{1-p}{2}} - \delta_{a-b-\frac{1+p}{2}} - \delta_{a-b+\frac{1+p}{2}} \tag{4.10}$$

where

$$\delta_a \equiv \delta_{a,0 \bmod n}\,. \tag{4.11}$$

Note that the arguments of delta functions in (4.10) are always integers, thanks to the requirement that p is odd.

The above quadratic form has a natural interpretation in K-theory, which easily generalizes to all values of d and finite groups Γ acting linearly on $\mathbb{C}^d$. Let $R(\Gamma)$ denote the representation ring of Γ. As we have seen, it is necessary to choose a lift of Γ to the spin group, so we assume Γ is a discrete subgroup of the spin group. Since Γ acts on the spin bundle we can define the equivariant index, and hence a bilinear pairing on $K^0_\Gamma(X)$:

$$(E,F) \to Ind_\Gamma(\not{D}_{E^*\otimes F}) \in R(\Gamma) \tag{4.12}$$

Composing with the projection $\pi_* : R(\Gamma) \to \mathbb{Z}$ defined by the projection to the trivial representation we obtain a bilinear pairing on $K^0_\Gamma(X)$ with values in $\mathbb{Z}$. This is the natural pairing on branes defined by elements of equivariant K-theory. Now let us specialize to $X = \mathbb{C}^d$. Quite generally, $K_\Gamma(X) \cong K_\Gamma(pt) \cong R(\Gamma)$ whenever X is equivariantly contractible to a point [24]. Evaluating the topological index we obtain the pairing on fractional branes thought of as elements of $K_\Gamma(\mathbb{C}^d) \cong R(\Gamma)$. This pairing is given by

$$(\rho_1, \rho_2) = \pi_*\Big(\bar{\rho}_1 \otimes \rho_2 \otimes (S^+ - S^-)\Big)\,. \tag{4.13}$$

Here $S^\pm$ are the chiral spin representations on $\mathbb{C}^d$ regarded as Γ-modules. This pairing is precisely that computed above using D-brane techniques.

5. Finding the change of basis

5.1 Statement of the main result

The intersection matrix I_{ab} should be interpreted as the intersection form evaluated on the elements of the equivariant K-theory of the orbifold. We denote this by

$$I_{ab} = (e_a, e_b)\,. \tag{5.1}$$

We will demonstrate the existence of an *integral* invertible linear transformation

$$f_a = U_a{}^b e_b \tag{5.2}$$

[4]The situation with type-0 is more subtle. The closed string sector of type-0 theory is invariant under $p \to p \pm n$; however the orbifold action in the open string sector does change, hence the theory is again defined by specifying p mod $2n$.

which block-diagonalizes I

$$F = -UIU^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C' \end{pmatrix} \tag{5.3}$$

In (5.3) the first row and the first column of zeroes correspond to the D0 brane, which is free to move off the orbifold fixed point. The next r rows and columns correspond to matrix elements between the special representations, $e_{p_r=1}, e_{p_{r-1}}, \ldots, e_{p_1}$. Moreover, C is the generalized $r \times r$ Cartan matrix (1.2) which describes the geometric intersection of the exceptional divisors in the minimal Hirzebruch-Jung resolution. We will find that U has the simple form

$$U = \begin{pmatrix} 1 & 1_{1\times r} & 1_{1\times r'} \\ 0_{r\times 1} & 1_{r\times r} & u \\ 0_{r'\times 1} & 0_{r'\times r} & 1_{r'\times r'} \end{pmatrix} \tag{5.4}$$

where u is an $r \times r'$ matrix of integers.

5.2 Examples

Here we provide some examples which illustrate the general claim outlined above. The first example is rather trivial, $n(p) = n(-1)$ which gives rise to the supersymmetric orbifold. The continued fraction is $[2^{n-1}]$, so all generators survive the GSO projection, as expected. As $r = n-1$, all e_i correspond to special representations, and no additional change of basis is required.

The next example is $n(p) = n(1-n)$. (Note that n must be even, so that $p = 1-n$ is odd.) The action on worldsheet bosons here is the same as in the $n(1)$ orbifold, but the latter suffers from the merciless GSO projection which eliminates the generator of the chiral ring. The continued fraction is $[n]$. It is not difficult to show that

$$u = \left(2, 3, \ldots, \frac{n}{2}, -\frac{n}{2}+1, -\frac{n}{2}+2, \ldots, -1\right) \tag{5.5}$$

is the solution with all required properties.

Our next example is $n(p) = (3m+1)(-3)$. The continued fraction is $[2^{m-1}, 4]$. The solution is given by

$$u = \begin{pmatrix} 2 & -1 & 1 & -1 & 1 & -1 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}. \tag{5.6}$$

5.3 The proof of the main result

Let us omit the first row and the first column, which corresponds to restricting to the subspace spanned by the fractional branes, e_i, $i = 1, \ldots, n-1$. (The first row and column contain the overlaps of e_i with $D0 - \sum_{i=1}^{n-1} e_i$, where $D0$ stands for the D0 brane which can move off the orbifold singularity.) In the following the overlap matrices will be always

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restricted to this subspace, and will be denoted by the same letters as their unrestricted counterparts above. We will prove that for the restricted matrix I, written in the basis

$$\{e_{p_1}, \ldots, e_{p_r}, \{e_\nu\}\} \tag{5.7}$$

where ν labels Coulomb representations, i.e. runs over $2, \ldots, n-1$ with p_α excluded, there exists a matrix U of the form

$$U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \tag{5.8}$$

with u being an $r \times (n-r-1)$ matrix of integers, such that

$$-UIU^T = \begin{pmatrix} C & 0 \\ 0 & C' \end{pmatrix} \tag{5.9}$$

where C is the generalized Cartan matrix. We will actually prove the inverse of (5.9), which is equivalent, since all the matrices there are non-singular. Denote -

$$-I^{-1} = \begin{pmatrix} \tilde{c} & \tilde{x} \\ \tilde{x}^T & \tilde{c}' \end{pmatrix}. \tag{5.10}$$

Using

$$U^{-1} = (\,1 \quad -u0 \quad 1\,) \tag{5.11}$$

and a similar formula for the transpose, we can write $-(U^T)^{-1}I^{-1}U^{-1}$ as

$$-(U^T)^{-1}I^{-1}U^{-1} = \begin{pmatrix} \tilde{c} & -\tilde{c}u + \tilde{x} \\ -u^T\tilde{c} + \tilde{x}^T & -u^T(-\tilde{c}u + \tilde{x}) - \tilde{x}^T u + \tilde{c}' \end{pmatrix}. \tag{5.12}$$

To prove (5.9) (or rather its inverse) it is sufficient to prove two statements. First, $\tilde{c} = C^{-1}$, or

$$(-I^{-1})_{p_\alpha p_\beta} = (C^{-1})_{\alpha\beta} = \begin{cases} \frac{1}{n} q_\alpha p_\beta & 1 \le \alpha \le \beta \le r \\ \frac{1}{n} p_\alpha q_\beta & 1 \le \beta \le \alpha \le r\,. \end{cases} \tag{5.13}$$

where we used an expression for C^{-1} derived in [2]. Second,

$$u = C\tilde{x} \tag{5.14}$$

is a matrix of integers. Let us start with the first statement.

We want to compute $(-I^{-1})_{jk}$. Define

$$\phi_{jk} = \frac{\exp\left(\frac{2\pi i jk}{n}\right)}{\sqrt{n}} = \frac{w^{jk}}{\sqrt{n}}, \qquad j,k = 1, \ldots, n-1\,. \tag{5.15}$$

and

$$\Sigma = \mathrm{diag}\{\sigma_j\}, \qquad \sigma_j = 4 \sin\left(\frac{\pi j}{n}\right) \sin\left(\frac{\pi j p}{n}\right). \tag{5.16}$$

In matrix notation the inverse of (4.9) reads

$$I^{-1} = \phi^{-1}\Sigma^{-1}(\bar{\phi})^{-1}, \tag{5.17}$$

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where bar denotes complex conjugation. The matrix ϕ is not hard to invert; the result is

$$(\phi^{-1})_{jk} = \bar{\phi}_{jk} - \frac{1}{\sqrt{n}}\,. \tag{5.18}$$

This can be verified by making use of the identities

$$\phi_{jk}\bar{\phi}_{kl} = \delta_{jl} - \frac{1}{n}\,. \tag{5.19}$$

and

$$\sum_{k=1}^{n-1} \phi_{jk} = -\frac{1}{\sqrt{n}}\,. \tag{5.20}$$

Combining (5.18) and (5.17), we have

$$(-I^{-1})_{jk} = N_{jk}(n;p_1) \tag{5.21}$$

where we define

$$N_{jk}(n;p_1) = \frac{1}{n}\sum_{s=1}^{n-1} \frac{(e^{\frac{-2\pi ijs}{n}}-1)(e^{\frac{2\pi iks}{n}}-1)}{4\sin\left(\frac{\pi s}{n}\right)\sin\left(\frac{\pi(n-p_1)s}{n}\right)}\,. \tag{5.22}$$

To deal with this formula we draw inspiration from appendix A of [3]. Consider the meromorphic function

$$f_{jk}(z) = -\frac{(e^{-2\pi ijz}-1)(e^{2\pi ikz}-1)}{4\sin(\pi z)\sin(\pi p_1 z)\sin(\pi nz)}\,. \tag{5.23}$$

We consider contour integrals of this function around the contour given by $\mathcal{C} = C_1 - C_2 - C_3 + C_4$ where C_1 runs along $1-\epsilon+iy$, $-\Lambda \le y \le \Lambda$, C_2 runs along $x+i\Lambda$, $-\epsilon \le x \le 1-\epsilon$, C_3 runs along $-\epsilon+iy$, $-\Lambda \le y \le \Lambda$, and C_4 runs along $x-i\Lambda$, $-\epsilon \le x \le 1-\epsilon$. Here $\epsilon < 1/n$.

Note that by contour deformation, and evaluation of residues we learn that

$$\frac{1}{2i}\oint_{\mathcal{C}} f_{jk}(z)dz = -\frac{jk}{p_1 n} + N_{jk}(n;p_1) + N_{jk}(p_1;n)\,. \tag{5.24}$$

On the other hand, we can explicitly evaluate the integrals along the countour $C_1 - C_2 - C_3 + C_4$ as follows. Note that $f_{jk}(z+1) = f(z)$ as long as $p = p_1 - n$ is an odd number, which is precisely the condition for type-II theory to be well defined. Hence, the integrals over C_1 and $-C_3$ cancel each other. The integrals along C_2, C_4 are not zero in general. However, along C_4, we may expand f_{jk} in inverse powers of $\xi := e^{i\pi z}$, which has a large absolute magnitude. The result is that the contribution of the segment C_4 is

$$\frac{1}{2i}\int_{-\epsilon}^{1-\epsilon} dx \xi^{2k-1-p_1-n} \frac{(1-\xi^{-2j})(1-\xi^{-2k})}{(1-\xi^{-2})(1-\xi^{-2p_1})(1-\xi^{-2n})}\,. \tag{5.25}$$

A similar formula holds for C_2. Note that for p odd, the expansion is in powers of ξ^{-2} and hence the integral vanishes unless there is a term $\sim \xi^0$. We therefore define some coefficients

$$\frac{(1-\xi^{-2j})(1-\xi^{-2k})}{(1-\xi^{-2})(1-\xi^{-2p_1})(1-\xi^{-2n})} := \sum_{\ell=0}^{\infty} A_\ell \xi^{-\ell} \tag{5.26}$$

and define moreover $A_\ell = 0$ when $\ell < 0$. Note that these coefficients are *integers*. In terms of these integers we may write

$$\frac{1}{2i}\oint_{C_1-C_2-C_3+C_4} dz f_{jk}(z) = -A_{(2j-1-p_1-n)} - A_{(2k-1-p_1-n)}\,. \tag{5.27}$$

Note, especially, that the integral (5.27) vanishes if $j,k < \frac{p_1+n+1}{2}$.

Comparing (5.24) to (5.27), we obtain a recursion formula for $N_{jk}(n;p_1)$. We now combine this recursion formula with the observation that

$$N_{jk}(p_1;n) = -N_{jk}(p_1;p_2) \tag{5.28}$$

where p_2 is defined to be $p_2 = a_1 p_1 - n$, and which holds when a_1 is an even integer. As long as j,k satisfy the relevant bound so that (5.27) vanishes we can now write

$$N_{jk}(n;p_1) = -N_{jk}(p_1;n) + \frac{jk}{p_1 n} = N_{jk}(p_1;p_2) + \frac{jk}{p_2 p_1} + \frac{jk}{p_1 n} = \cdots \tag{5.29}$$

where we keep getting $N_{jk}(p_\alpha;p_{\alpha+1})$ with higher and higher α's. For this construction to work, all a_α's must be even integers. Recall that this is precisely the condition for the chiral ring generators to survive the GSO projection. If either j or k is equal to p_α, the process terminates when we reach $N_{jk}(p_\alpha;p_{\alpha+1}) = 0$.

Consider $\tilde{c}_{\alpha\beta} = I^{-1}_{j=p_\alpha,k=p_\beta}$. Suppose $\alpha \le \beta$ which implies $j = p_\alpha \ge k = p_\beta$. The recursion process in (5.29) terminates at $N_{jk}(p_\alpha;p_{\alpha+1}) = 0$. There are no additional integer contributions since $j,k \le p_1 < \frac{n+p_1+1}{2}$. That is, we have

$$\tilde{c}_{\alpha\beta} = p_\beta p_\alpha \sum_{\gamma=1}^{\alpha} \frac{1}{p_\gamma p_{\gamma-1}}\,. \tag{5.30}$$

The desired equality, (5.13), follows from

$$p_\alpha \sum_{\gamma=1}^{\alpha} \frac{1}{p_\gamma p_{\gamma-1}} = \frac{q_\alpha}{n}\,. \tag{5.31}$$

This can be proven by induction. Define

$$H_\alpha = p_\alpha \sum_{\gamma=1}^{\alpha} \frac{1}{p_\gamma p_{\gamma-1}} \tag{5.32}$$

and note that $H_1 = 1/n = q_1/n$, $H_2 = a_1/n = q_2/n$. Then it is not hard to prove that the induction hypothesis

$$H_\alpha = a_{\alpha-1} H_{\alpha-1} - H_{\alpha-2} \tag{5.33}$$

implies

$$H_{\alpha+1} = a_\alpha H_\alpha - H_{\alpha-1}\,. \tag{5.34}$$

To summarize,

$$\tilde{c}_{\alpha\beta} = \frac{q_\alpha p_\beta}{n}\,, \qquad \alpha \le \beta\,. \tag{5.35}$$

When $\alpha \geq \beta$, the sequence in (5.29) terminates at $N_{jk}(p_\beta; p_{\beta+1})$ and so α and β in (5.35) are interchanged. This concludes the proof of (5.13).

To prove that $u = C\tilde{x}$ is a matrix of integers, write this as

$$u_{\alpha\nu} = C_{\alpha\beta}\tilde{x}_{\beta\nu} = -\tilde{x}_{\alpha-1,\nu} + a_\alpha \tilde{x}_{\alpha,\nu} - \tilde{x}_{\alpha+1,\nu} \tag{5.36}$$

where index ν labels Coulomb representations, i.e. runs over $2, \ldots, n-1$, with $p_1, \ldots, p_r$ excluded. The computation of $\tilde{x}_{\alpha,\nu} = N_{p_\alpha,\nu}(n; p_1)$ proceeds via reduction, just as in the case of $\tilde{c}_{\alpha\beta}$. The process in (5.29) has additional integer contributions from the coefficients A_ℓ in (5.27) but still terminates at $N_{p_\alpha,\nu}(p_\alpha; p_{\alpha+1}) = 0$. However it is no longer true that $\nu < \frac{p_\alpha + p_{\alpha+1}}{2}$, and hence some of the coefficients A_ℓ will be nonzero. Nevertheless, we can say that

$$\tilde{x}_{\beta,\nu} = \frac{q_\beta \cdot \nu}{n} + b_{\beta,\nu}\,, \tag{5.37}$$

where $b_{\beta,\nu}$ is a sum of coefficients of the type A_ℓ. In particular, they are integers. Substituting this into (5.36) gives

$$u_{\alpha\nu} = -b_{\alpha-1,\nu} + a_\alpha b_{\alpha,\nu} - b_{\alpha+1,\nu} \in \mathbb{Z}\,. \tag{5.38}$$

This completes the proof.

6. Discussion of some loose ends

6.1 Coupling to the bulk graviton

The mass of a fractional brane can be measured by the overlap of the corresponding boundary state with the bulk graviton and is given by

$$m_{1/n} = \frac{m_0}{n}\,, \tag{6.1}$$

where $m_0 = 1/g_s l_s$ is the mass of a D0 brane. Higgs branch branes which wrap minimal cycles in the resolved geometry come from the linear combination of fractional branes determined by (5.2). As explained in the previous section, to each Higgs representation of the orbifold group $\mathbb{Z}_n$, labeled by p_α, we can associate a linear combination of fractional branes e_i,

$$h_\alpha = e_{p_\alpha} + \sum_\nu u_\alpha^{\ \nu} e_\nu\,. \tag{6.2}$$

According to (6.1), the mass of the corresponding boundary state is given by

$$\frac{m_\alpha}{m_{1/n}} = 1 + \sum_\nu u_\alpha^{\ \nu}\,. \tag{6.3}$$

The computation of the sum in (6.3) is similar in spirit to the computations performed in the previous section. We start by noting

$$\sum_{i=1}^{n}\sum_{\beta=1}^{r} C_{\alpha\beta}(-I^{-1})_{p_\beta i} = \sum_{\gamma=1}^{r}\sum_{\beta=1}^{r} C_{\alpha\beta}(C^{-1})_{\beta\gamma} + \sum_\nu u_\alpha^{\ \nu}\,, \tag{6.4}$$

where we used (5.10), (5.13) and (5.14). Substituting (6.4) into (6.3), we obtain

$$\frac{m_\alpha}{m_{1/n}} = \sum_\beta C_{\alpha\beta} \sum_{i=1}^{n} (-I^{-1})_{p_\beta i} \tag{6.5}$$

where we used $\sum_{\beta=1}^{r} C_{\alpha\beta}(C^{-1})_{\beta\gamma} = \delta_{\alpha\gamma}$. To compute $\sum_{i=1}^{n}(-I^{-1})_{p_\beta i}$ we perform summation over i in the expression for I^{-1} [equations (5.21) and (5.22)]; the result is

$$\sum_{i=1}^{n} (-I^{-1})_{p_\beta i} = n M_\beta(n;p) \tag{6.6}$$

where we define

$$M_\beta(n;p_1) = -\frac{1}{n} \sum_{s=1}^{n-1} \frac{(e^{\frac{-2\pi i p_\beta s}{n}} - 1)}{4\sin\left(\frac{\pi s}{n}\right)\sin\left(\frac{\pi(n-p_1)s}{n}\right)}\,. \tag{6.7}$$

This quantity is computed similarly to $N_{jk}(n;p_1)$- we again consider integral along $\mathcal{C}$ of the meromorphic function

$$h_\beta(z) = \frac{(e^{-2\pi i p_\beta z} - 1)}{4\sin(\pi z)\sin(\pi p_1 z)\sin(\pi n z)}\,. \tag{6.8}$$

The analog of (5.24) is now

$$0 = -\frac{p_\beta^2}{2p_1 n} + M_\beta(n;p_1) + M_\beta(p_1;n)\,, \tag{6.9}$$

where the first term in the right-hand side comes from the residue at $z = 0$, and the total integral is zero because $2p_\beta < n + p_1 + 1$. Next we can perform the recursion process, thanks to the identity $M_\beta(n;p_1) = -M_\beta(p_1;p_2)$. The recursion terminates at p_β; combining everything together we have

$$\frac{m_\alpha}{m_{1/n}} = \sum_{\beta=1}^{r} \frac{C_{\alpha\beta} p_\beta q_\beta}{2}\,. \tag{6.10}$$

This formula evaluates to positive integers for a variety of (n,p); however we have not bothered to prove this in general.

6.2 Other values of (n,p) and nonminimal resolutions

The proof of the previous section makes heavy use of the hypothesis that a_α are all even, and this is the origin of our restriction on the values of n,p mentioned in the introduction. What happens when some of the partial quotients a_α are odd? In this case some of the generators are projected out in the type-II string and the above story becomes more complicated. In string theory language, the geometry cannot be resolved by turning on localized tachyons. We expect that one can perform a partial block diagonalization involving fractional branes at remaining singularities and Higgs branch branes associated to smooth cycles. But we have not carried this out.

Similar remarks apply to nonminimal resolutions. In this case one of the partial quotients will be equal to one, which is odd. For example, if we blow up a cycle that corresponds to a generator

$$W = W_j W_{j+1}\,, \tag{6.11}$$

then, as shown in [3, 2], the continued fraction $n/p_1 = [a_1, \ldots, a_r]$ is replaced by

$$\frac{n}{p_1} = [a_1, \ldots, a_j + 1, 1, a_{j+1} + 1, \ldots, a_r]\,. \tag{6.12}$$

Hence, we see that only a subset of orbifold resolutions can be realized within type-II theory. This is again a consequence of the chiral GSO projection.

6.3 Type-0 strings

It is natural to ask what happens in type-0 string theory where the diagonal GSO acts trivially on the chiral ring. Consider the fractional brane overlap matrix in the type-0 case. The number of D-branes is now doubled. However the boundary states of type-0 are related to the boundary states of type-II in a simple way. Namely, the so-called "electric" (v) and "magnetic" (o) states of type-0 have the form [23]

$$\begin{aligned} |a, v\rangle &= \sum_s B_a^s \frac{1}{\sqrt{2}} \left(|s, NS, +\rangle\rangle + |s, R, +\rangle\rangle\right); \\ |a, o\rangle &= \sum_s B_a^s \frac{1}{\sqrt{2}} \left(-|s, NS, -\rangle\rangle + |s, R, -\rangle\rangle\right) \end{aligned} \tag{6.13}$$

where $|s, NS, \pm\rangle\rangle$, $|s, R, \pm\rangle\rangle$ are Ishibashi states that couple to closed strings in the s twisted sector and B_a^s are coefficients which can be determined from the Cardy condition [23]. Type-II boundary states are of the form

$$|a, II\rangle = \sum_s B_a^s \frac{1}{2} \left(|s, NS, +\rangle\rangle - |s, NS, -\rangle\rangle + |s, R, +\rangle\rangle + |s, R, -\rangle\rangle\right). \tag{6.14}$$

Comparing (6.13) and (6.14), we infer that the type-0 intersection form is given by

$$I_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{6.15}$$

where I is the type-II intersection form. Physically, chiral fermions only exist when branes of opposite type intersection

It is worthwhile remarking that the above doubling of branes, together with the intersection form (6.15) is extremely natural from the K-theoretic viewpoint. Type-0 is an orbifold of the type-II string by $(-1)^{F_{st}}$, where F_{st} is spacetime fermion number [25]. If a group G acts trivially on a space X then there is a natural isomorphism $K_G(X) \cong K(X) \otimes R(G)$ given by

$$E \to \oplus_a \mathrm{Hom}(E_a, E) \otimes \rho_a \tag{6.16}$$

where $E_a = X \times \rho_a$ is a trivial G bundle associated to the irrep ρ_a of G. If we regard the type-0 string as an orbifold of the type-II string by $\mathbb{Z}_2$ with generator σ acting trivially on

X then we see that $K_{\mathbb{Z}_2}(X) \cong K(X) \oplus K(X) \otimes \eta$ where η is the nontrivial irrep of $\mathbb{Z}_2$. The lift of σ to the spin group acts as -1; this is what we mean by an orbifold by $(-1)^{F_{st}}$. Thus, the K-theory orientation class is $(S^+ - S^-) \otimes \eta$. This has an effect on the bilinear pairing. Denoting by $(\cdot,\cdot)$ the tensor product of the natural pairings on $K(X) \otimes R(\mathbb{Z}_2)$ the index pairing on $K_{\mathbb{Z}_2}(X)$ is

$$Ind(E_1, E_2) = (E_1, E_2 \otimes \eta) \tag{6.17}$$

for $E_1, E_2 \in K(X) \otimes R(\mathbb{Z}_2)$. This is precisely the intersection matrix (6.15).

In the type-0 case the intersection form (6.15) is composed out of the type-II intersection matrices. One peculiar feature of this theory is that the closed string sector is insensitive to the shift $p \to p + n$. Hence, we expect the same Hirzebruch-Jung space as a resolution of the orbifolds $\mathbb{C}^2/\mathbb{Z}_{n(p)}$ and $\mathbb{C}^2/\mathbb{Z}_{n(p+n)}$. However the intersection form I is certainly different in these two cases. Consider, for example, the supersymmetric orbifold with $p = -1$, where the fractional branes simply evolve into the branes wrapping cycles in the resolved ALE space. No additional change of basis is required in this case, as the intersection form $I_0(p = -1)$ has the form (6.15) with I being the Cartan matrix of the supersymmetric orbifold. But the intersection form $I_0(p = n - 1)$ is very different. In the simple example $n = 3$ it is possible to bring it to the supersymmetric form by a change of basis, $I_0(p = -1) = UI_0(p = n - 1)U^T$, with U being an invertible matrix of integers. However it is rather difficult to construct such a change of basis in general. This should be contrasted with the type-II case, where the change of basis has a relatively simple form and the fractional branes that correspond to Higgs representations naturally become the branes wrapping cycles in the resolved geometry.

6.4 Intersection matrix for Coulomb branch branes

So far, we have focused on the fractional branes and the Higgs branch branes. It is natural to ask what can be said about the Coulomb branch branes. In particular, given the change of basis (1.4) we expect the pairing on Coulomb branch branes to be given by the matrix C' in (5.3).

In general, the intersection form for branes in Landau-Ginzburg models is difficult to compute, although some nontrivial progress has been made [26]–[29]. For this reason, we restrict our discussion to the case of $d = 1$, i.e. $\mathbb{C}/\mathbb{Z}_n$ orbifolds. In this case, there is one Higgs branch brane, which may be identified with the $D0$ brane on the resolution of the space $\mathbb{C}/\mathbb{Z}_n$, and $(n - 1)$ Coulomb branch branes described by a Landau-Ginzburg model with superpotential $W(X)$ given by an order n polynomial [3].

When the superpotential is $W(X) = X^n$ we can think of the branes in terms of those of the $\mathcal{N} = 2$ minimal conformal field theory. The latter have a simple geometrical description described in [30], which, moreover, is nicely compatible with the intersection form on branes. According to [30], the branes may be pictured as oriented straight lines in the unit disk (= parafermion target space) joining special points on the boundary of the disk. These special points are at integral multiples of the angle $\phi_0 = \frac{\pi}{n}$. In the type-0 theory the branes separate into two types - those joining even multiples of ϕ_0 and those joining odd multiples of ϕ_0. In the language of [30], these are the Cardy states $|\hat{j}, \hat{n}, \hat{s}\rangle$ with $\hat{s} = \pm 1$ and $\hat{s} = 0, 2$, respectively, and they may be identified with the so-called "electric" and "magnetic" branes of the type-0 theory. As discussed in [30] a generating set of branes is given by the shortest chords. These are e_s^- joining $(2s-2)\phi_0$ to $2s\phi_0$, $s = 1, \ldots, n$ while e_s^+ joins $(2s-1)\phi_0$ to $(2s+1)\phi_0$, $s = 1, \ldots, n$. As explained in [30], two concatenated chords with the same orientation are classically unstable to form a shorter chord. Therefore, the state $e_1^\pm + \cdots + e_n^\pm$, which may be pictured as a polygonal ring, is unstable to shrinking into the disk. We interpret this brane as the Higgs branch brane of the spacetime, and the relation $e_1^\pm + \cdots + e_n^\pm = 0$ of [30] as the relation defining the space of Coulomb branch branes. The intersection form can be computed from the boundary state. It turns out that $(e_s^-, e_{s'}^-) = (e_s^+, e_{s'}^+) = 0$, while the intersection form of odd-branes with even-branes is simply the geometrical oriented intersection number. In particular, $(e_s^-, e_{s'}^+) = \delta_{s,s'}$. Thus the intersection form is simply

$$\begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix}. \tag{6.18}$$

Note that the change of basis $e_s^+ \to e_{s+1}^+$ on the odd branes brings the intersection matrix to the form

$$\begin{pmatrix} \mathcal{S} & 0_{n\times n} \\ 0_{n\times n} & 1_{n\times n} \end{pmatrix} \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix} \begin{pmatrix} \mathcal{S}^{tr} & 0_{n\times n} \\ 0_{n\times n} & 1_{n\times n} \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{S} \\ -\mathcal{S}^{tr} & 0 \end{pmatrix} \tag{6.19}$$

so we neatly reproduce the intersection form of the fractional branes of type-0. For the type-II projection we note that spectral flow from the NS to the R sector takes $e_s^- \to e_{s+1}^+$ so the type-II branes have a basis $e_s = e_s^- + e_{s+1}^+$. The intersection form is $(e_s, e_{s'}) = -\delta_{s+1,s'} + \delta_{s,s'+1}$, in perfect conformity to that of the fractional branes computed in section 4.

When the Landau-Ginzburg polynomial is deformed to $W = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ the critical points separate. Let us adopt the heuristic picture advocated in [3]. Then, associated with each critical point is a "universe" with its own Higgs branch brane. If the critical point is not Morse, then it has a collection of Coulomb branes associated to it. Since we have the unit matrix in the 12 block in (6.18) there is no problem splitting the branes into collections for each critical point. Thus, for finite RG flow the intersection form (6.18) is preserved. For infinite RG flow, the Witten index changes, as expected.

6.5 Other open problems

The present note raises some further interesting questions for the future.

One evident open problem is the generalization of our discussion to $d > 2$. We expect that similar techniques will apply but many - possibly very nontrivial - details remain to be worked out.

It would be quite illuminating to construct boundary states directly within the framework of the gauged linear sigma model and literally follow the RG evolution of the boundary state. It would also be nice to understand the relation of the quantum McKay correspondence to the discussion of "missing D-brane charges" found in [31].

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Finally, understanding real time dynamics of tachyon condensation as opposed to studying the RG flow is an important subject in need of clarification. See [32, 33] for recent work on this topic. The generalization to $d = 2$ appears to be quite nontrivial. It would be good to understand the fate of D-branes in these time-dependent backgrounds.

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